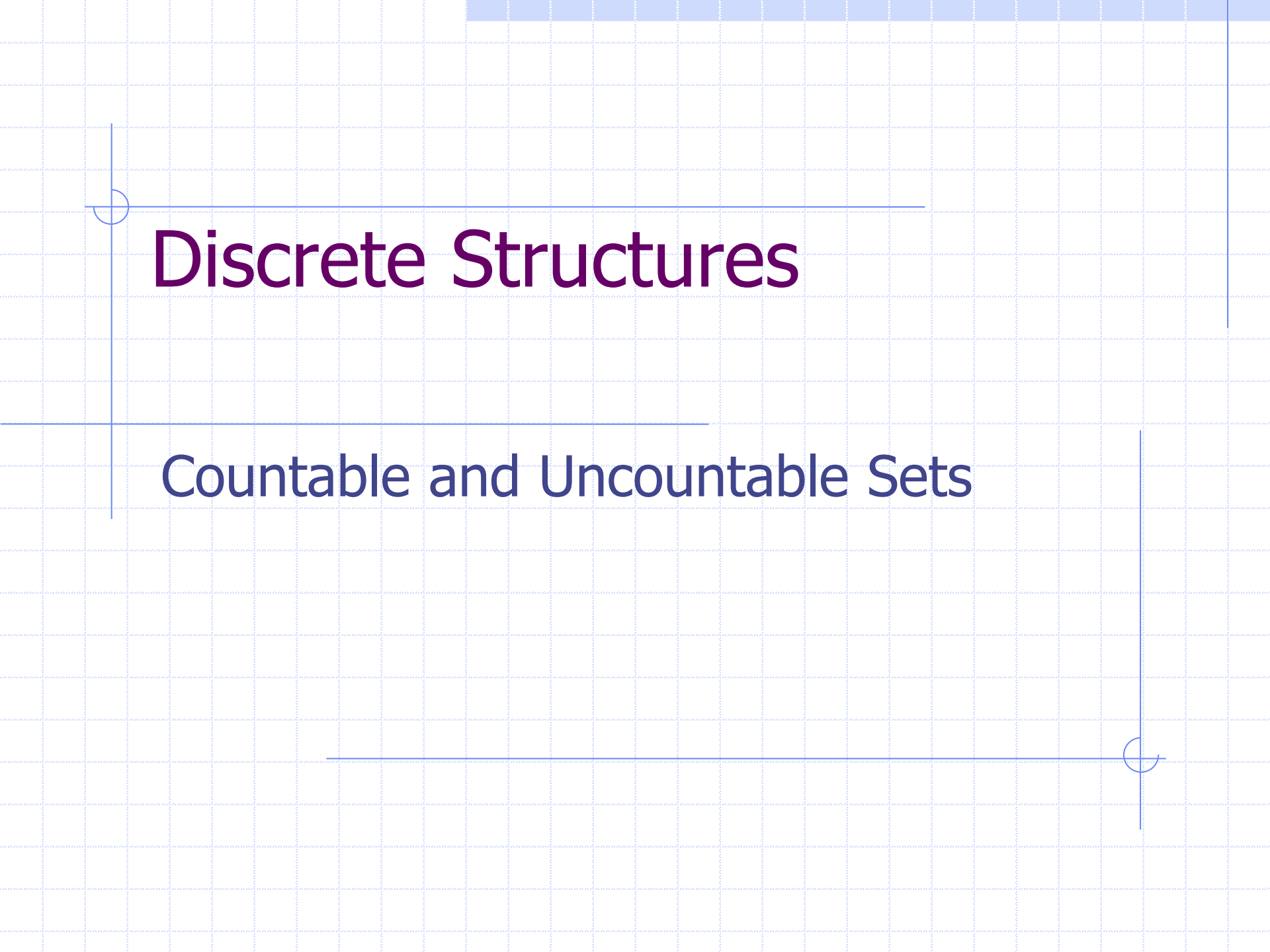




Discrete Structures

Countable and Uncountable Sets

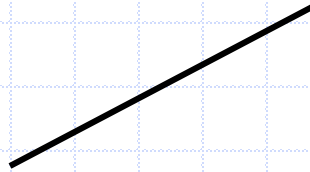
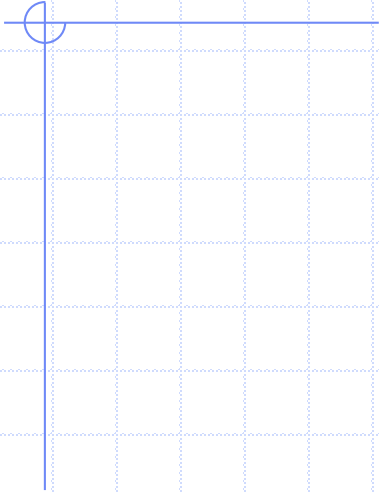
Cardinality

Def.: The cardinality of a set A is the number of elements in the set and denoted by $|A|$

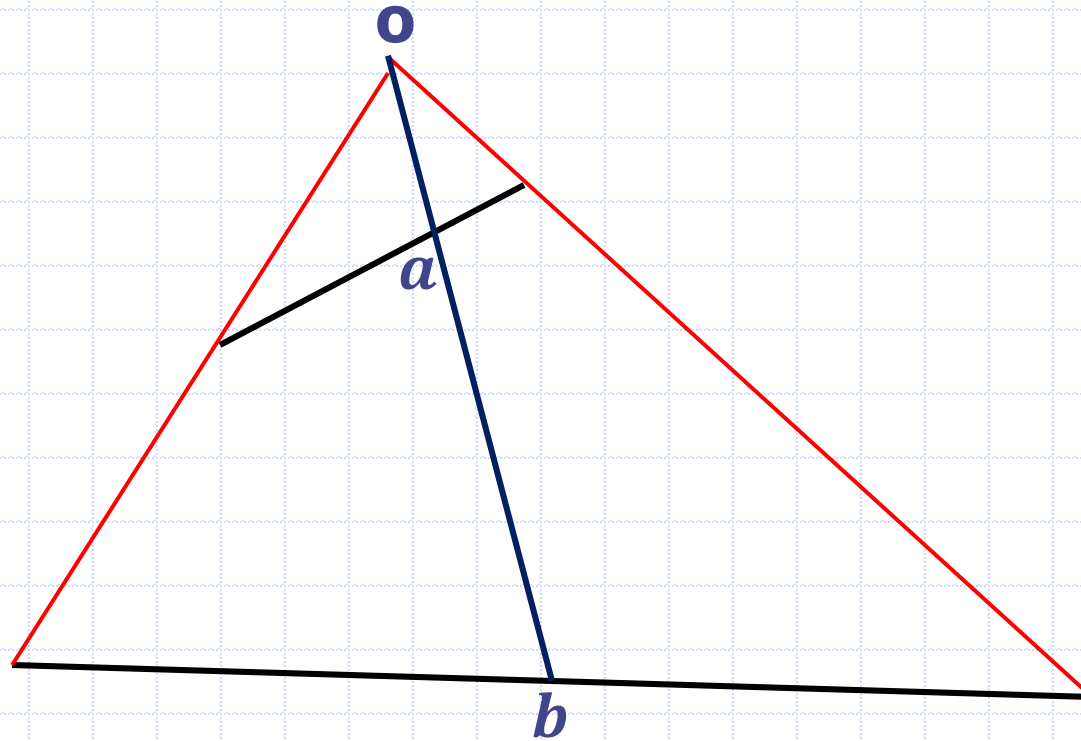
Def.: Let A and B be two sets.

- $|A| = |B|$ iff there is a one-to-one and onto function (bijection) from A to B
- $|A| \leq |B|$ iff there is a one-to-one function from A to B
- If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$
(Schröder-Bernstein theorem: No need to know the proof)

Cardinality



Cardinality



Countable Sets and Uncountable Sets

Def.: Set A is **countable** if it is finite or if it has the same cardinality as the set of positive integers. Otherwise it is **uncountable**.

Examples:

- ♦ Infinite Countable Sets: $\mathbb{N}, \mathbb{Z}^-, \mathbb{Z}$
- ♦ Infinite Uncountable Sets: $\mathbb{R}, \mathbb{R}^+, \mathbb{R}^-$

Countable Sets and Uncountable Sets

How do you demonstrate that a set is countable?

Suppose A is a set. If there is a **one-to-one and onto function** $f: A \rightarrow \mathbb{N}$, then A is countable. Recall,

one-to-one means $\forall x \forall y (f(x) = f(y) \rightarrow x = y)$

onto means $\forall y \exists x (f(x) = y)$

Countable Sets and Uncountable Sets

Theorem:

Set $\{ x \mid x \text{ is an odd positive integer} \}$ is countable.

Proof: We need a one-to-one correspondence between this set and $\mathbb{N}$

1, 3, 5, 7, 9, ... corresponds to $a_1, a_2, a_3, a_4, a_5 \dots$

We could also consider $f(2n-1) = n$ from the set of odd positive integers to $\mathbb{N}$.

Countable Sets and Uncountable Sets

Theorem:

The set $\mathbf{Z}$ is countable.

Proof: List them like this: 0, -1, 1, -2, 2, -3, 3, -4, 4 ...

Which corresponds to $a_1, a_2, a_3, a_4, a_5, a_6 \dots$

What we've actually done is given the one-to-one correspondence between all integers and the positive integers, i.e., the mapping from $\mathbf{Z}$ to $\mathbf{N}$

What about an expression for this?

$f(n) = 2n+1$ when n is non-negative

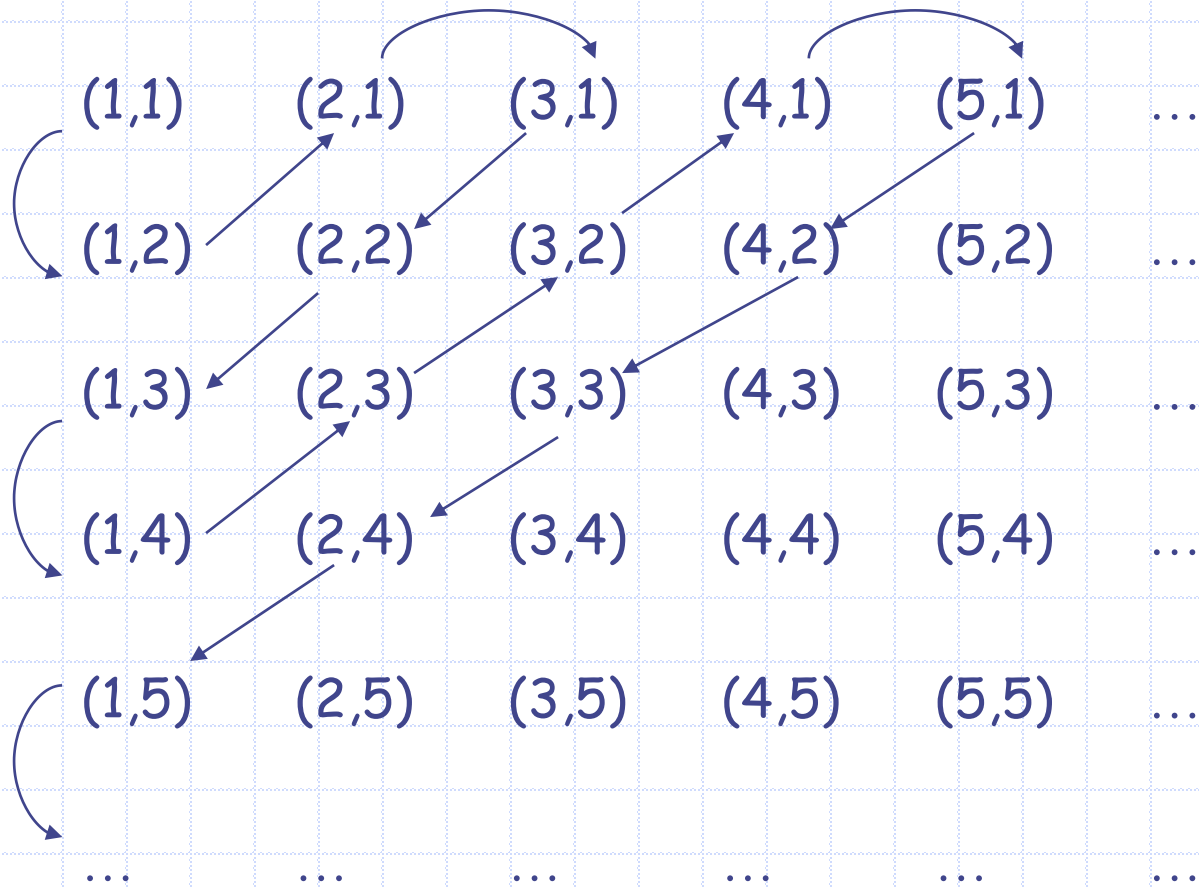
$f(n) = -2n$ when n is negative

Countability

Theorem: The set P of all ordered pairs of positive integers (n, m) is countable.

Proof: Can we find a one-to-one and onto function from P to $\mathbb{N}$?

Countability



$$f(m, n) = ?$$

Countability

Easy one-to-one function but not onto from P to N

$$f(m, n) = 2^m 3^n$$

Countability

Easy one-to-one function but not onto from P to N

$$f(m, n) = 2^m 3^n$$

$$|P| \leq |N|$$

It is clear that $|N| \leq |P|$, then $|N| = |P|$

Countability

If A and B are countable sets, then the following are countable sets

- Any subset of A
- $A \cup B$
- $A \cap B$
- $A \times B$

Note that $P = N \times N$ and rational numbers can be seen as a subset of P

Uncountable sets

Theorem: The set of real numbers is uncountable.

If a subset of a set is uncountable, then the set is uncountable.
The cardinality of a subset is at least as large as the cardinality of the entire set.

It is enough to prove that there is a subset of $\mathbb{R}$ that is uncountable

Theorem: The open interval of real numbers $[0,1) = \{r \in \mathbb{R} \mid 0 \leq r < 1\}$ is uncountable.

Proof by contradiction using the *Cantor diagonalization argument* (Cantor, 1879)

Uncountable Sets: R

Proof using *diagonalization*: Suppose $\mathbb{R}$ is countable (then any subset say $[0,1)$ is also countable). So, we can list them: $r_1, r_2, r_3, \dots$ where

$$r_1 = 0.d_{11}d_{12}d_{13}d_{14}\dots \quad \text{the } d_{ij} \text{ are digits 0-9}$$

$$r_2 = 0.d_{21}d_{22}d_{23}d_{24}\dots$$

$$r_3 = 0.d_{31}d_{32}d_{33}d_{34}\dots$$

$$r_4 = 0.d_{41}d_{42}d_{43}d_{44}\dots$$

etc.

Now let $r = 0.d_1d_2d_3d_4\dots$ where $d_i = 4$ if $d_{ii} \neq 4$
 $d_i = 5$ if $d_{ii} = 4$

But r is not equal to any of the items in the list so it's missing from the list so we can't list them after all.

r differs from r_i in the i^{th} position, for all i . So, our assumption that we could list them all is incorrect.